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Theory of External Two-State Markov Noise in the Presence of Internal Fluctuations

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A theory is presented to take into account internal fluctuations in the study of stochastically driven systems. Internal fluctuations are modeled by a master equation in which external noise is introduced. External noise is modeled by a two-state Markov process. A unified theory of internal and external fluctuations is described in terms of an effective integrodifferential master equation or its equivalent generating function representation. Two examples for which exact analytical results can be obtained are presented. A discussion of the white noise limit of the theory is also given.

KEY WORDS: Fluctuations; external noise; master equation; finite size effects.

1. INTRODUCTION

In the study of open systems it is convenient to distinguish between external and internal fluctuations. This difference depends, of course, in what one chooses to define as the system. In practice, the difference between the system and its external parameters is clearly established in each particular case. The external parameters are determined by the environment of the system: boundary conditions, applied fields, etc. Internal fluctuations are those self-originated in the system. Their study is an important well-known part of statistical mechanics both in equilibrium and far from equilibrium. Generally speaking they reflect the statistical nature of a macroscopic description. They are associated with the large number of degrees of freedom averaged out in such description. An important fact about internal fluctuations is that they scale with the system size. Therefore, they vanish in the thermodynamic limit, except at a critical point where long-range order is established.

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External fluctuations are those present in the system when this is subject to an external noise. Therefore, they are not "self-originated." External noise is here understood as the existence of external parameters which do not take a fixed value but random values. External noise can be originated by an intrinsic natural randomness of the environment of the system. It can also be intentionally applied in a given experimental setup: a control parameter of the system is forced to take random values with a well-defined prescription. In this last situation external noise plays the role of an external field driving the system. The analysis of this situation parallels other studies in which the behavior of a system is considered when driven away from equilibrium. This is, for example, the case of the study of critical dynamics away from equilibrium.[1] External noise situations have been recently studied from the theoretical and experimental point of view.[2,3] Different parametrizations of external noise[4-8] and nonlinear noise coupling[9] have been analyzed. Systems on which the effect of external noise has been considered include, among others, chemical reactions,[10] electrical circuits,[11-14] liquid crystals,[9,15-17] hydrodynamical systems,[18,19] laser and optical systems,[20-23] plasmas,[24] nuclear reactors,[25] etc. The theoretical studies of external noise situations made up to now rely generally on the assumption of negligible internal fluctuations. The rationale for this procedure is that for a macroscopic system, external noise, if present, will dominate the internal fluctuations because the latter scale with the system size. On these grounds, external noise has been usually considered as a stochastic process to be implemented in deterministic equations which describe the behavior of a macroscopic system. Our purpose in this paper is to present a unified treatment in which external as well as internal fluctuations are simultaneously considered. The relevance of this approach is twofold. First, from a first principles point of view it is desirable to have a framework in which internal and external fluctuations can be jointly analized. The hypothesis of negligible internal fluctuations can then be justified by taking the thermodynamic limit in the presence of external noise. Moreover, in such framework, there exists the possibility of finding novel features associated with the interplay of internal and external fluctuations. These effects could not be analyzed when internal or external fluctuations are neglected. Secondly, from a practical point of view, this treatment provides with a method for calculating finite size effects in stochastically driven systems. For macroscopic finite systems these effects will be corrections to the predicted behavior in the absence of internal fluctuations. This aspect can be particularly important in thermodynamically small systems like lasers in some circumstances or electronic midrodevices. For very small systems, internal fluctuations and external fluctuations may become comparable.

In this paper, and as first step towards more complicated situations, we have considered simple ways of incorporating internal and external fluctuations into a unified description. We consider spatially homogeneous systems described by a single relevant variable. Internal fluctuations are modeled by a one-step markovian master equation. This model of internal fluctuations has been used in the literature for chemical reactions[26–28] laser systems,[29] tunnel diode circuits,[30] masers,[31] radioactive decay,[32] nuclear reactors,[33] biological systems,[34] etc. External noise is modeled by fluctuating parameters described by a two-state markov process (dichotomic noise).

The main points of our approach can be summarized as follows. From the master equation describing internal fluctuations we write the associated differential equation for the generating function. In this equation external noise is introduced by means of a fluctuating parameter. This leads to a stochastic partial differential equation. The average of this equation over the realizations of the external noise gives an equation for an effective generating function. This function gives a complete unified description of the statistical properties of the system. An equivalent representation is given by an integrodifferential master equation for a probability density which incorporates internal and external fluctuations. In the thermodynamic limit this integrodifferential equation reduces to the one which describes external fluctuations in the standard approach in which internal fluctuations are neglected. Finite size effects can be calculated from either the effective generating function or the probability density.

The outline of the paper is the following: Section 2 contains the general theory. In Section 2.1 we summarize the standard independent approaches to internal fluctuations via a master equation and to external noise via a stochastic differential equation with a dichotomic noise. In Sections 2.2 and 2.3 we present the unified theory of internal fluctuations and external dichotomic noise. Section 2.2 is devoted to the generating function representation and Section 2.3 to the master equation representation. In Section 2.4 we discuss an alternative method which makes use of the Poisson representation of the master equation.[35] In Section 3 we present two illustrative examples. We close the paper with a discussion in Section 4. In particular we discuss the difficulties associated with the white noise limit[36] of the results obtained for a dichotomic noise.

2. GENERAL THEORY

2.1. Standard Approaches

Internal fluctuations in an homogeneous system have been studied in a great number of systems by means of a Markovian master equation. This master equation has the general form

$$\frac{\partial P(N,t)}{\partial t} = \sum_{L=1}^{\infty} [W(N,N-L;t)\,P(N-L,t) + W(N,N+L;t)\,P(N+L,t)] - \sum_{L=1}^{\infty} [W(N+L,N;t) + W(N-L,N;t)]\,P(N,t) \quad (2.1)$$

$P(N, t)$ is, for example for a chemical model, the probability of having N particles of a given reactant at time t and $W(N, N \pm L; t)$ is the transition probability at time t from a state with $N \pm L$ particles to a state with N particles. We only consider here one-step master equations in which L only takes the value 1. We also introduce the following notation for the one-step transition probabilities[37]:

$$Q(N,t) \equiv W(N+1,N;t) \quad (2.2)$$

$$R(N,t) \equiv W(N-1,N;t) \quad (2.3)$$

With this notation (2.1) becomes for one-step processes,

$$\frac{\partial P(N,t)}{\partial t} = Q(N-1,t)\,P(N-1,t) + R(N+1,t)\,P(N+1,t) - [Q(N,t) + R(N,t)]\,P(N,t) \equiv \Gamma\left(N, \frac{\partial}{\partial N}, t\right) P(N,t) \quad (2.4)$$

where we have introduced the operator Γ

$$\Gamma\left(N, \frac{\partial}{\partial N}, t\right) = (e^{-\partial/\partial N} - 1)\,Q(N,t) + (e^{\partial/\partial N} - 1)\,R(N,t) \quad (2.5)$$

The condition of detailed balance is always satisfied for a one-step master equation. As a consequence[27] a general formula for the stationary solution of (2.4) is known.

The transition probabilities are in general assumed to be extensive quantities, that is, proportional to the system size. For a chemical model this is the volume V of the system. Defining a density

$$x = N/V \quad (2.6)$$

we have

$$Q(N) = Vq(x), \qquad R(N) = Vr(x) \quad (2.7)$$

The functions $q(x)$ and $r(x)$ can also have contributions proportional to V^{-n}.

An interesting and useful representation of the master equation is given by the generating function $F(s, t)$. This is defined by[26]

$$F(s, t) = \sum_{N=0}^{\infty} s^N P(N, t) \tag{2.8}$$

This function satisfies the differential equation[34]

$$\frac{\partial F(s, t)}{\partial t} = \left[(s-1)\, Q\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s} - 1\right) R\left(s\frac{\partial}{\partial s}\right)\right] F(s, t) \tag{2.9}$$

with a boundary condition which guarantees the normalization of $P(N, t)$:

$$F(s = 1, t) = 1 \tag{2.10}$$

The generating function contains all the statistical information on the system. The probability distribution and its moments are, respectively, obtained as

$$P(N, t) = \frac{1}{N!} \frac{\partial^N}{\partial s^N} F(s, t)\Big|_{s=0} \tag{2.11}$$

$$\langle N^m \rangle = \sum_{N=0}^{\infty} N^m P(N, t) = \left(s\frac{\partial}{\partial s}\right)^m F(s, t)\Big|_{s=1} \tag{2.12}$$

where $\langle \cdots \rangle$ indicates an average taken with the probability distribution $P(N, t)$.

In the variable $z = s - 1$, the generating function admits a Taylor expansion in the form

$$F(z, t) = \sum_{m=0}^{\infty} b_m z^m \tag{2.13}$$

where (2.10) implies $b_0 = 1$. The factorial moments $\langle \Omega_m \rangle$ of $P(N, t)$ defined by

$$\langle \Omega_m \rangle = \langle N(N-1) \cdots (N - m + 1) \rangle \tag{2.14}$$

are given by

$$\langle \Omega_m \rangle = \frac{\partial^m F(s, t)}{\partial s^m}\Big|_{s=1} = \frac{\partial^m F(z, t)}{\partial z^m}\Big|_{z=0} = m!\, b_m \tag{2.15}$$

The fluctuations modeled by the master equation scale with the system size and vanish in the thermodynamic limit $N \to \infty$, $V \to \infty$, $x = N/V$ finite.

In this limit the master equation description reduces to a deterministic description[38-40] defined by an evolution equation for x:

$$\frac{dx}{dt} = q(x) - r(x) \tag{2.16}$$

In this equation an eventual dependence of $q(x)$ and $r(x)$ in terms proportional to V^{-n} is assumed to be dropped.

The deterministic equation (2.16) is the usual starting point in the studies of systems subject to an external noise influence.[2,3] Under the assumption of negligible internal fluctuations, the existence of external noise is modeled by substituting a parameter α in (2.16) by a stochastic process as $\alpha = \bar{\alpha} + \xi(t)$. Here $\bar{\alpha}$ is the man value of α and $\xi(t)$ is its fluctuation. This procedure transforms (2.16) into a stochastic differential equation. For simplicity we assume that the parameter α is only included in $q(x)$ and that $q(x)$ is a linear function of α: $q(x) = q_0^0(x) + \alpha q_1(x)$. Replacing α by $\bar{\alpha} + \xi(t)$ we have that $q(x) \to q_0(x) + \xi(t)\, q_1(x)$, where $q_0(x)$ is $q(x)$ with $\alpha \to \bar{\alpha}$. In this case the stochastic differential equation is

$$\dot{x} = q_0(x) - r(x) + q_1(x)\, \xi(t) \tag{2.17}$$

A common way of dealing with the stochastic process defined by (2.17) is to look for the equation obeyed by the probability density of the process $P(x, t)$. This equation depends on the nature of $\xi(t)$. Here we assume that $\xi(t)$ is a two-state markov process which takes values $\pm\Delta$ and has correlation time λ^{-1}:

$$\overline{\xi(t)} = 0, \qquad \overline{\xi(t)\, \xi(t')} = \Delta^2 \exp(-\lambda\, |t - t'|) \tag{2.18}$$

where $\overline{\cdots}$ indicates an average over the realizations of the external noise $\xi(t)$. The dichotomic Markov noise $\xi(t)$ is hardly a good model of natural environmental noise. Nevertheless, it can be easily applied to a system in laboratory conditions. This is a well-controled situation to study the cooperative response to a random perturbation. Specific examples are considered in Ref. 2.

Under these assumptions the probability density $P(x, t)$ obeys the following integrodifferential equation:

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\,[q_0(x) - r(x)]\, P(x,t) \\ &\quad + \Delta^2 \frac{\partial}{\partial x} q_1(x) \int_0^t \exp\left\{-\left(\lambda + \frac{\partial}{\partial x}\,[q_0(x) - r(x)]\right)(t - t')\right\} \\ &\quad \times \frac{\partial}{\partial x} q_1(x)\, P(x, t')\, dt'\end{aligned} \tag{2.19}$$

or the two linear partial differential equations

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} [q_0(x) - r(x)] P(x, t) - \frac{\partial}{\partial x} q_1(x) P_1(x, t) \tag{2.20}$$

$$\frac{\partial}{\partial t} P_1(x, t) = -\lambda P_1(x, t) - \frac{\partial}{\partial x} [q_0(x) - r(x)] P(x, t) - \Delta^2 \frac{\partial}{\partial x} q_1(x) P(x, t) \tag{2.21}$$

where

$$P_1(x, t) = \overline{\xi(t)\, \delta(x - x(t))}, \qquad P(x, t) = \overline{\delta(x - x(t))} \tag{2.22}$$

The stationary solution of (2.19) is

$$P_0(x) = C \frac{q_1(x)}{\Delta^2 q_1^2(x) - [q_0(x) - r(x)]^2} \times \exp\left\{\int^x dx' \frac{\lambda(q_0(x') - r(x'))}{\Delta^2 q_1^2(x') - [q_0(x') - r(x')]^2}\right\} \tag{2.23}$$

The details of the derivation of Eqs. (2.19)–(2.23) can be found in Refs. 5 and 41.

An important thing to keep in mind is that the fluctuating parameters in $q(x)$ are assumed to be parameters determined from conditions external to the system under consideration. This is the meaning of their "external noise" character. In particular they cannot have an explicit dependence on the volume of the system V. As a consequence, the noise intensity Δ^2 is an intensive quantity and it remains constant in the thermodynamic limit. In a chemical model the fluctuating parameter can be, for example, the concentration of a reactant which is kept constant on the average from outside of the system. Another example is the light intensity in a photochemical reaction.[10]

2.2. External Noise in the Equation for the Generating Function

In this section and in Section 2.3 we develop a formalism for the joint study of internal an external fluctuations. The presence of external noise is considered in the equations which take into account internal fluctuations of the system. This is done by the same method that external noise is modeled in the absence of internal fluctuations. That is, we substitute the fixed value of an external parameter by a stochastic process. This can be done in the master equation (2.4) or in the equation for the generating function (2.9). We first consider external noise in the equation for the generating function. The discussion for the master equation is given in Section 2.3.

With the same simplifying assumptions that in (2.17) the substitution of α by $\bar{\alpha} + \xi(t)$ leads to a stochastic transition probability of the form

$$Q(N, t) = Q_0(N) + Q_1(N)\,\xi(t) \tag{2.24}$$

The requirement of positivity of $Q(N)$ implies that $Q_0(N) - Q_1(N)\Delta \geqslant 0$ [$Q_1(N)$ and $\bar{\alpha}$ are positive quantities]. This requirement on the value of Δ is satisfied when the external parameter α does not change its sign when it fluctuates. Otherwise, the starting master equation (2.4) becomes, at least in principle, meaningless.

Substituting (2.24) in (2.9) we obtain

$$\begin{aligned}\frac{\partial F(s,t)}{\partial t} &= \left[(s-1)\,Q_0\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s} - 1\right) R\left(s\frac{\partial}{\partial s}\right)\right] F(s,t) \\ &\quad + (s-1)\,\xi(t)\,Q_1\left(s\frac{\partial}{\partial s}\right) F(s,t)\end{aligned} \tag{2.25}$$

This is a stochastic partial differential equation for $F(s, t)$. The generating function $F(s, t)$ is now a functional of $\xi(t)$. Averaging this equation over the realizations of $\xi(t)$ we obtain an equation for an effective generating function $\bar{F}(s, t)$ defined as the average of $F(s, t)$ over the realizations of $\xi(t)$:

$$\bar{F}(s,t) \equiv \overline{F(s,t)} \tag{2.26}$$

This equation is

$$\begin{aligned}\frac{\partial \bar{F}(s,t)}{\partial t} &= \left[(s-1)\,Q_0\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s} - 1\right) R\left(s\frac{\partial}{\partial s}\right)\right] \bar{F}(s,t) \\ &\quad + (s-1)\,Q_1\left(s\frac{\partial}{\partial s}\right) F_1(s,t)\end{aligned} \tag{2.27}$$

where

$$F_1(s,t) \equiv \overline{\xi(t)\,F(s,t)} \tag{2.28}$$

To obtain a closed set of equations we now look for the equation satisfied by $F_1(s, t)$. This is obtained using the "diferentiation formula" of Shapiro and Loginov[42]

$$\frac{\partial}{\partial t}\,\overline{\xi(t)\,F(s,t)} = -\lambda\,\overline{\xi(t)\,F(s,t)} + \overline{\xi(t)\,\frac{\partial}{\partial t} F(s,t)} \tag{2.29}$$

Thus,

$$\frac{\partial}{\partial t} F_1(s,t) = -\lambda F_1(s,t) + \left[(s-1)\, Q_0 \left(s\frac{\partial}{\partial s} \right) + \left(\frac{1}{s} - 1 \right) R \left(s\frac{\partial}{\partial s} \right) \right] F_1(s,t)$$
$$+ \Delta^2 (s-1)\, Q_1 \left(s\frac{\partial}{\partial s} \right) \bar{F}(s,t) \tag{2.30}$$

Equations (2.27) and (2.30) form the set of closed equations. Integrating formally (2.30) and substituting in (2.27) we obtain the following integrodifferential equation for $\bar{F}(s,t)$:

$$\frac{\partial}{\partial t} \bar{F}(s,t) = \left[(s-1)\, Q_0 \left(s\frac{\partial}{\partial s} \right) + \left(\frac{1}{s} - 1 \right) R \left(s\frac{\partial}{\partial s} \right) \right] \bar{F}(s,t)$$
$$+ \Delta^2 (s-1)\, Q_1 \left(s\frac{\partial}{\partial s} \right) \int_0^t \exp \left(- \left\{ \lambda + \left[(s-1)\, Q_0 \left(s\frac{\partial}{\partial s} \right) \right. \right. \right.$$
$$\left. \left. \left. + \left(\frac{1}{s} - 1 \right) R \left(s\frac{\partial}{\partial s} \right) \right] \right\} (t-t') \right)$$
$$\times (s-1)\, Q_1 \left(s\frac{\partial}{\partial s} \right) \bar{F}(s,t')\,dt' \tag{2.31}$$

Here we have set $F_1(s,0) = 0$, which implies statistical independence of $\xi(t)$ and $F(s,t)$ at $t = 0$.

Equation (2.31) gives a complete description of the statistical properties of the system when both internal and external fluctuations are taken into account.

2.3. External Noise in the Master Equation

We now consider the master equation (2.4) in the presence of external noise. When we substitute (2.24) in (2.4), $P(N,t)$ becomes a functional of $\xi(t)$. We then define $\bar{P}(N,t)$ and $P_1(N,t)$ by

$$\bar{P}(N,t) = \overline{P(N,t)}, \qquad P_1(N,t) = \overline{\xi(t)\, P(N,t)} \tag{2.32}$$

which are related to $\bar{F}(s,t)$ and $F_1(s,t)$ by

$$\bar{F}(s,t) = \sum_{N=0}^{\infty} s^N \bar{P}(N,t), \qquad F_1(s,t) = \sum_{N=0}^{\infty} s^N P_1(N,t) \tag{2.33}$$

Inverting the steps which lead to (2.9) from (2.4) one can obtain the equations for $\bar{P}(N,t)$ and $P_1(N,t)$ from equations (2.27) and (2.30)

$$\frac{\partial \bar{P}(N,t)}{\partial t} = \Gamma_0 \bar{P}(N,t) + \Gamma_{1,Q} P_1(N,t) \tag{2.34}$$

$$\frac{\partial P_1(N,t)}{\partial t} = -\lambda P_1(N,t) + \Gamma_0 P_1(N,t) + \Delta^2 \Gamma_{1,Q} \bar{P}(N,t) \tag{2.35}$$

where Γ_0 is defined as in (2.4) with $Q(N)$ replaced by $Q_0(N)$

$$\Gamma_{1,Q} = (e^{-\partial/\partial N} - 1)\, Q_1(N) \tag{2.36}$$

Integrating formally (2.35) with $P_1(N,0) = 0$ and substituting in (2.34) we obtain an integrodifferential equation satisfied by $\bar{P}(N,t)$:

$$\frac{\partial}{\partial t} \bar{P}(N,t) = \Gamma_0 \bar{P}(N,t) + \Delta^2 \Gamma_{1,Q} \int_0^t \exp[-(\lambda + \Gamma_0)(t-t')]\, \Gamma_{1,Q} \bar{P}(N,t')\, dt' \tag{2.37}$$

This equation gives a complete description of the problem. It is equivalent to Eq. (2.31).

It is also interesting to consider a joint probability $P(N, \pm\Delta; t)$ for the two processes N and $\xi(t)$. This is defined as the probability of having N particles and the external noise the value $\pm\Delta$ at time t.

From the definition (2.32) it is easy to see that

$$\bar{P}(N,t) = P(N,\Delta;t) + P(N,-\Delta;t) \tag{2.38}$$

$$P_1(N,t) = \Delta[P(N,\Delta;t) - P(N,-\Delta;t)] \tag{2.39}$$

Inverting these algebraic equations and using (2.34) and (2.35) we obtain the following equations for $P(N, \pm\Delta; t)$:

$$\begin{aligned} \frac{\partial P(N,\Delta;t)}{\partial t} &= [Q_0(N-1) + \Delta Q_1(N-1)]\, P(N-1,\Delta;t) + R(N+1)\, P(N+1,\Delta;t) \\ &\quad - [Q_0(N) + \Delta Q_1(N) + R(N) + \lambda/2]\, P(N,\Delta;t) + (\lambda/2)\, P(N,-\Delta;t) \end{aligned} \tag{2.40}$$

$$\begin{aligned} \frac{\partial P(N,-\Delta;t)}{\partial t} &= [Q_0(N-1) - \Delta Q_1(N-1)]\, P(N-1,-\Delta;t) + R(N+1)\, P(N+1,-\Delta;t) \\ &\quad - [Q_0(N) - \Delta Q_1(N) + \lambda/2]\, P(N,-\Delta;t) + (\lambda/2)\, P(N,\Delta;t) \end{aligned} \tag{2.41}$$

Equations (2.40) and (2.41) constitute a master equation for the two variables N and $\xi(t)$. In these equations we find again the requirement $Q_0 - Q_1\Delta \geqslant 0$ to guarantee the positivity of the transition probabilities.

In the thermodynamic limit $V \to \infty$, $N \to \infty$, $N/V = x$ internal fluctuations dissapear and we recover the standard description of external noise: in this limit

$$\Gamma_0 = -\frac{\partial}{\partial x}[q_0(x) - r(x)], \qquad \Gamma_{1,Q} = -\frac{\partial}{\partial x} q_1(x) \tag{2.42}$$

and (2.37) reduces to (2.19). A systematic expansion in powers of V^{-1} of (2.37) provides a useful method to calculate finite size effects in a stochastically driven system. From such an expansion it is easy to see that in general there will be a coupling of internal and external fluctuations: there are terms proportional to $\Delta^2 V^{-n}$. These terms vanish in the thermodynamic limit and also for a finite system in the absence of external noise.

2.4. Use of the Poisson Representation

The Poisson representation of the master equation[35] is a useful method to study internal fluctuations. In this representation the statistical properties are calculated from a stochastic differential equation for a continuous variable instead of using the discrete master equation. This gives a formal similarity with the starting point of the standard approach to external noise. With this motivation we study in this section our unified theory of internal and external fluctuations making use of the Poissson representation of the master equation.

The Poisson representation of $P(N, t)$ is defined introducing a quasiprobability $f(a, t)$[35]

$$P(N, t) = \int da\, e^{-a} \frac{a^N}{N!} f(a, t) \tag{2.43}$$

An important property of $f(a, t)$ is that the factorial moments (2.14) of $P(N, t)$ are just the moments of $f(a, t)$:

$$\langle \Omega_m \rangle = \int da\, a^m f(a, t) \tag{2.44}$$

The differential equation satisfied by $f(a, t)$ is obtained as follows. Without loss of generality we assume that the transition probabilities are proportional to factorial moments:

$$Q(N) = \alpha V^{-(m-1)} N(N-1) \cdots (N-m+1) \tag{2.45}$$

$$R(N) = \beta V^{-(l-1)} N(N-1) \cdots (N-l+1) \tag{2.46}$$

In (2.45) we have explicitly taken into account the extensivity property of $Q(N)$ and $R(N)$, and α and β are volume-independent constants. Substituting (2.45) and (2.46) in (2.4), using (2.43), and integrating by parts we obtain that the different terms of the master equation are transformed as

$$Q(N-1)\,P(N-1,t)-Q(N)\,P(N,t)\to -\frac{\alpha}{V^{m-1}}\cdot\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}f(a,t) \tag{2.47}$$

$$R(N+1)\,P(N+1,t)-R(N)\,P(N,t)\to \frac{\beta}{V^{l-1}}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^{l-1}a^{l}f(a,t) \tag{2.48}$$

The equation satisfied by $f(a,t)$ is then

$$\frac{\partial f(a,t)}{\partial t}=-\frac{\partial}{\partial a}\left[\frac{\alpha}{V^{m-1}}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}-\frac{\beta}{V^{l-1}}\left(1-\frac{\partial}{\partial a}\right)^{l-1}a^{l}\right]f(a,t) \tag{2.49}$$

In the case in which $m\leqslant 1$ and $l\leqslant 2$ this is a Fokker–Planck-type equation which can be solved by standard methods.

External noise can be taken into account by substituting a parameter in (2.49) by a stochastic processes. For example we replace α by $\bar{\alpha}+\xi(t)$. This leads to a stochastic partial differential equation for $f(a,t)$, which becomes now a functional of $\xi(t)$:

$$\begin{aligned}\frac{\partial f(a,t)}{\partial t}=&-\frac{\partial}{\partial a}\left[\frac{\bar{\alpha}}{V^{m-1}}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}-\frac{\beta}{V^{l-1}}\left(1-\frac{\partial}{\partial a}\right)^{l-1}a^{l}\right]f(a,t)\\&-\frac{\xi(t)}{V^{m-1}}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}f(a,t)\end{aligned} \tag{2.50}$$

We now follow the same mathematical steps than in the case of the generating function of Section 2.2. We define $\bar{f}(a,t)$ as the average of $f(a,t)$ over the realizations of $\xi(t)$ and $f_1(a,t)$ as

$$f_1(a,t)=\overline{\xi(t)\,f(a,t)} \tag{2.51}$$

These two functions obey a closed set of equations:

$$\begin{aligned}\frac{\partial \bar{f}(a,t)}{\partial t}=&-\frac{\partial}{\partial a}\left[\frac{\bar{\alpha}}{V^{m-1}}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}-\frac{\beta}{V^{l-1}}\left(1-\frac{\partial}{\partial a}\right)^{l-1}a^{l}\right]\bar{f}(a,t)\\&-\frac{1}{V^{m-1}}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^{m}a^{m}f_1(a,t)\end{aligned} \tag{2.52}$$

$$\frac{\partial f_1(a,t)}{\partial t} = -\lambda f_1(a,t) - \frac{\partial}{\partial a}\left[\frac{\bar{\alpha}}{V^{m-1}}\left(1-\frac{\partial}{\partial a}\right)^m a^m - \frac{\beta}{V^{l-1}}\left(1-\frac{\partial}{\partial a}\right)^{l-1} a^l\right] f_1(a,t) - \frac{\Delta^2}{V^{m-1}}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^m a^m \bar{f}(a,t) \tag{2.53}$$

An example in which the stationary solution of (2.52) and (2.53) can be explicitly calculated is given in the next section. These equations have a formal similarity with Eqs. (2.27)–(2.30) for the generating function. The two sets of equations are partial differential equations for functions of a continuous variable. As a consequence it is in general easier to deal with them than with Eqs. (2.34) and (2.35). An advantage of the Poisson representation over the generating function method is that $\bar{f}(a,t)$ is more directly connected with the probability density of the process. In general $\bar{P}(N,t)$ is the Poisson transform of $\bar{f}(a,t)$. This connection becomes particularly clear in the thermodynamic limit: introducing an intensive variable $y = a/V$ we have for a factorial moment

$$\left\langle x\left(x-\frac{1}{V}\right)\cdots\left(x-\frac{m-1}{V}\right)\right\rangle = \int dy\, y^m V\bar{f}(y,t) \tag{2.54}$$

Here $\overline{\langle\cdots\rangle}$ indicates the average over both internal and external fluctuations. Equation (2.54) shows that in the thermodynamic limit $V\bar{f}(y,t)$ becomes the probability density $\bar{P}(x,t)$. Equations (2.52) and (2.53) can be rewritten for the variable y. Eliminating $f_1(y,t)$ we obtain an integrodifferential equation for $\bar{f}(y,t)$

$$\begin{aligned}\frac{\partial}{\partial t}\bar{f}(y,t) &= -\frac{\partial}{\partial y}\left[\bar{\alpha}\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^m y^m - \beta\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^{l-1} y^l\right]\bar{f}(y,t)\\ &\quad + \Delta^2\frac{\partial}{\partial y}\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^m y^m \int_0^t \exp\left(-\left\{\lambda + \frac{\partial}{\partial y}\left[\bar{\alpha}\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^m y^m\right.\right.\right.\\ &\quad \left.\left.\left. - \beta\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^{l-1} y^l\right]\right\}(t-t')\right)\frac{\partial}{\partial y}\left(1-\frac{1}{V}\frac{\partial}{\partial y}\right)^m\\ &\quad \times y^m\bar{f}(y,t')\,dt'\end{aligned} \tag{2.55}$$

In the thermodynamic limit (2.55) reduces to (2.19). No such easy direct connection exists between (2.31) and (2.19). For a finite V, (2.55) is a good starting point to study finite size effects in a perturbation around the thermodynamic limit.

3. EXAMPLES

We present in this section two illustrative examples in which we apply the general techniques of the previous section. The first example admits an exact time-dependent solution for the effective generating function $\bar{F}(s, t)$. In a particular limit the relevance of the positivity requirement of the transition probability becomes apparent. In the second example we obtain exact expressions for the steady state effective generating function and probability distribution. In both examples the relative fluctuations have a contribution originated by the external noise and another one due to internal fluctuations which vanishes in the thermodynamic limit.

Example 1. The first example is a Poisson counting process defined by the transition probabilities

$$Q(N) = \alpha V, \qquad R(N) = 0 \tag{3.1}$$

This model has no steady-state solution. External noise is introduced through fluctuations in the parameter α so that $Q_0(N) = \bar{\alpha} V$, $Q_1(N) = V$. Equations (2.27) and (2.30) become here

$$\frac{\partial \bar{F}}{\partial t} = (s-1)\,\bar{\alpha} V \bar{F} + (s-1)\,V F_1 \tag{3.2}$$

$$\frac{\partial F_1}{\partial t} = -\lambda F_1 + (s-1)\,\bar{\alpha} V F_1 + (s-1)\,V \Delta^2 \bar{F} \tag{3.3}$$

These equations have to be solved with the boundary condition (2.10) and the initial conditions $F_1(s, t=0) = 0$ and $F(s, t=0) = s^{N_0}$ $[\bar{P}(N, t=0) = \delta_{N,N_0}]$. Equations (3.2) and (3.3) are a set of linear equations with eigenvalues

$$X_{1,2} = \{2\bar{\alpha} V(s-1) - \lambda \pm [\lambda^2 + 4\Delta^2 V^2 (s-1)^2]^{1/2}\}/2 \tag{3.4}$$

Therefore

$$\bar{F}(s, t) = A(s)\exp(X_1 t) + B(s)\exp(X_2 t) \tag{3.5}$$

where

$$A(s) = \frac{s^{N_0}\{\lambda + [\lambda^2 + 4\Delta^2 V^2 (s-1)^2]^{1/2}\}}{2[\lambda^2 + 4\Delta^2 V^2 (s-1)^2]^{1/2}} \tag{3.6}$$

$$B(s) = \frac{s^{N_0}\{-\lambda + [\lambda^2 + 4\Delta^2 V^2 (s-1)^2]^{1/2}\}}{2[\lambda^2 + 4\Delta^2 V^2 (s-1)^2]^{1/2}} \tag{3.7}$$

The implications of the positivity requirement $Q_0(N) \geqslant Q_1(N)\Delta$ are clearly seen in a limiting case of this example. We consider a static limit for the noise in which its correlation time $\lambda^{-1} \to \infty$. In this limit

$$\bar{F}(s,t) = \frac{s^{N_0}}{2}\{\exp[V(s-1)(\bar{\alpha}+\Delta)t] + \exp[V(s-1)(\bar{\alpha}-\Delta)t]\} \quad (3.8)$$

Expanding in powers of s we immediately obtain from (2.11)

$$\bar{P}(N,t) = \frac{[V(\bar{\alpha}+\Delta)t]^N}{2N!}\exp[-V(\bar{\alpha}+\Delta)t] + \frac{[V(\bar{\alpha}-\Delta)t]^N}{2N!}\exp[-V(\bar{\alpha}-\Delta)t] \quad (3.9)$$

Here we have taken $N_0 = 0$. The probability distribution is therefore the sum of two Poisson distributions with mean values $V(\bar{\alpha}+\Delta)t$ and $V(\bar{\alpha}-\Delta)t$. In the light of (2.38) the two terms in (3.9) are naturally interpreted. We now clearly see that the requirement $Q_0(N) \geqslant Q_1(N)\Delta$ that is $\bar{\alpha} \geqslant \Delta$, guarantees the positivity of $\bar{P}(N,t)$. For $\Delta > \bar{\alpha}$, $P(N, -\Delta; t)$ becomes negative for N odd. This situation corresponds to values of Δ for which $\alpha = \bar{\alpha} + \xi(t)$ takes negative values. In this case the master equation (2.4) lacks physical meaning.

From (2.12) and (3.8) we obtain

$$\langle \bar{N} \rangle = V\bar{\alpha}t \quad (3.10)$$

$$\frac{\langle \bar{N}^2 \rangle - \langle \bar{N} \rangle^2}{\langle \bar{N} \rangle^2} = \frac{\Delta^2}{\bar{\alpha}^2} + \frac{1}{V\bar{\alpha}t} \quad (3.11)$$

The mean value is independent of Δ and the relative fluctuation has a contribution from the internal noise which remains finite in the thermodynamic limit and another one coming from the internal fluctuations which vanishes in this limit. The internal noise contribution also vanishes as $t \to \infty$.

Example 2. Our second example is defined by the following chemical reaction

$$B + X \underset{k'}{\overset{k}{\rightleftharpoons}} C$$

where the concentrations b and c of the reactants B and C are kept constant from outside of the system. This reaction is described by the master equation (2.4) with

$$Q(N) = Vk'c \equiv V\alpha \quad (3.12)$$

$$R(N) = kbN \equiv \beta N \quad (3.13)$$

The stationary solution of the master equation is a Poisson distribution

$$P_{st}(N) = \left(\frac{\alpha V}{\beta}\right)^N \frac{e^{-\alpha V/\beta}}{N!} \tag{3.14}$$

The first moment and the relative fluctuations are

$$\langle N \rangle = \frac{\alpha V}{\beta} \tag{3.15}$$

$$\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\beta}{\alpha V} \tag{3.16}$$

We model an external source of noise by fluctuations of the parameter α, $\alpha \to \bar{\alpha} + \xi(t)$. Equations (2.27) and (2.30) are in this case

$$\frac{\partial \bar{F}(s,t)}{\partial t} = (s-1)\, V\bar{\alpha}\bar{F}(s,t) - \beta(s-1)\frac{\partial}{\partial s}\bar{F}(s,t) + (s-1)\, VF_1(s,t) \tag{3.17}$$

$$\frac{\partial F_1(s,t)}{\partial t} = -\lambda F_1(s,t) + (s-1)\, V\bar{\alpha}F_1(s,t) - \beta(s-1)\frac{\partial}{\partial s}F_1(s,t)$$
$$+ \Delta^2 V(s-1)\; \bar{F}(s,t) \tag{3.18}$$

From these two equations we obtain for the steady-state effective generating function $\bar{F}_{st}$ a second-order differential equation

$$z\frac{d^2}{dz^2}\bar{F}_{st}(z) + \frac{\lambda - 2\bar{\alpha}Vz}{\beta}\frac{d}{dz}\bar{F}_{st}(z) - \frac{\lambda\bar{\alpha}V - (\Delta^2 - \bar{\alpha}^2)\,V^2 z}{\beta^2}\bar{F}_{st}(z) = 0 \tag{3.19}$$

where $z = s - 1$.

This equation admits a solution of the form (2.13) with the recurrence relation

$$b_1 = \frac{V\bar{\alpha}b_0}{\beta} = \frac{\bar{\alpha}V}{\beta} \tag{3.20}$$

$$n\left(n - 1 + \frac{\lambda}{\beta}\right) b_n - \frac{\bar{\alpha}V}{\beta}\left[2(n-1) + \frac{\lambda}{\beta}\right] b_{n-1} - \frac{V^2}{\beta}(\Delta^2 - \bar{\alpha}^2) b_{n-2} = 0 \tag{3.21}$$

From (2.15) and (3.20)–(3.21) we obtain the first moment and the relative fluctuation

$$\langle \bar{N} \rangle = b_1 = \frac{\bar{\alpha}V}{\beta} \tag{3.22}$$

$$\frac{\langle \bar{N}^2 \rangle - \langle \bar{N} \rangle^2}{\langle \bar{N} \rangle^2} = \frac{2b_2 + b_1 - b_1^2}{b_1^2} = \frac{\beta}{\bar{\alpha}V} + \frac{\beta\Delta^2}{\bar{\alpha}^2\lambda}\left(1 + \frac{\beta}{\lambda}\right)^{-1} \tag{3.23}$$

The mean value (3.22) coincides with the result (3.15) obtained in the absence of external noise. The reltive fluctuation has a new term with respect to Eq. (3.16). This new term is volume independent and therefore also present in studies in which internal fluctuations are neglected, that is in the thermodynamic limit $V \to \infty$.

A closed expression for the stationary probability density of this model is more easily obtained using the Poisson representation method of Section 2.4. In this example $f(a, t)$ satisfies Eq. (2.49) with $m = 0$, $l = 1$. Equations (2.52) and (2.53) become for this particular model

$$\frac{\partial \bar{f}(a, t)}{\partial t} = -\frac{\partial}{\partial a}(V\bar{a} - \beta a)\bar{f}(a, t) - V\frac{\partial}{\partial a}f_1(a, t) \tag{3.24}$$

$$\frac{\partial f_1(a, t)}{\partial t} = -\lambda f_1(a, t) - \frac{\partial}{\partial a}(V\bar{a} - \beta a) f_1(a, t) - V\Delta^2 \frac{\partial}{\partial a}\bar{f}(a, t) \tag{3.25}$$

The steady-state solution of (3.24) and (3.25) is given by

$$\bar{f}_{st}(a) = C[\Delta^2 V^2 - (\bar{a}V - \beta a)^2]^{\lambda/2\beta - 1} \tag{3.26}$$

where

$$C = \frac{\beta \Gamma(1/2 + \lambda/2\beta)}{(\Delta V)^{\lambda/\beta - 1}\Gamma(1/2)\,\Gamma(\lambda/2\beta)} \tag{3.27}$$

According to (2.43) the stationary probability distribution is

$$\begin{aligned}\bar{P}_{st}(N) &= C\int_{a_-}^{a_+} da\, e^{-a}\frac{a^N}{N!}[\Delta^2 V^2 - (\bar{a}V - \beta a)^2]^{\lambda/2\beta - 1} \\ &= C\frac{V^{N+1}}{N!}\int_{-\Delta}^{\Delta} dy\, e^{Vy/\beta}(\bar{a} - y)^N(\Delta^2 - y^2)^{\lambda/2\beta - 1}\end{aligned} \tag{3.28}$$

where

$$a_{\pm} = \frac{V}{\beta}(\bar{a} \pm \Delta) \tag{3.29}$$

Equation (3.26) or (3.28) gives a complete closed description of the statistical steady-state properties. The results (3.22) and (3.23) are easily recovered for Eq. (3.27) or (3.29), although it is clear that if one is only interested in the evaluation of the first moments a more straightforward method is the power series solution of Eq. (3.19).

The example we have considered here has also been studied in the white noise limit of $\xi(t)$.[36] A comparison of results is given in the next section.

4. DISCUSSION

In this paper we have presented a general formalism to deal in a unified way with internal and external fluctuations. This formalism leads to an equation (2.31) for a effective generating function which contains all the statistical information on the system. An equivalent representation is the one provided by the equation (2.37) for the effective probability distribution. The Poisson representation gives an alternative method to deal with the problem.

For a macroscopic system under the influence of an external source of noise, internal fluctuations have to be usually considered as finite size corrections to the dominant external fluctuations. In the examples that we have considered we have calculated explicitly these finite size corrections in the relative fluctuations. We have found that they are decoupled from the external noise contribution. Nevertheless in more general cases we expect to find a coupling of the two kinds of fluctuations which would give contributions proportional to both the external noise intensity and the inverse system size. This sort of finite size corrections do not exist for a finite system in the absence of external noise.

Our development has been based on the modeling of external noise by a two-state Markov process. In an earlier attempt to deal with this problem we assumed an external Gaussian white noise.[36,43] We wish now to discuss the relation between these two approaches. We first note that in the white noise limit of the dichotomic noise $\Delta \to \infty, \lambda \to \infty, \Delta^2/\lambda = D$ we recover the results of Refs. 36, 43. In this limit Eq. (2.37) for the effective probability density becomes

$$\frac{\partial \bar{P}(N,t)}{\partial t} = (\Gamma_0 + D\Gamma_{1,Q}^2)\, \bar{P}(N,t) \tag{4.1}$$

This is an effective master equation[36,43] with effective transition probabilities

$$\bar{W}(N, N-1) = Q_0(N-1) - DQ_1^2(N-1) - DQ_1(N)\, Q_1(N-1) \tag{4.2}$$

$$\bar{W}(N, N+1) = R(N+1) \tag{4.3}$$

$$\bar{W}(N, N-2) = DQ_1(N-2)\, Q_1(N-1) \tag{4.4}$$

The white noise limit of (2.31) gives the equation for the effective generating function associated with (4.1)

$$\frac{\partial \bar{F}(s,t)}{\partial t} = \left[(s-1)\, Q_0\left(s\,\frac{\partial}{\partial s}\right) + \left(\frac{1}{s}-1\right) R\left(s\frac{\partial}{\partial s}\right)\right] \bar{F}(s,t) + \left[D(s-1)\, Q_1\left(s\frac{\partial}{\partial s}\right)(s-1)\, Q_1\left(s\frac{\partial}{\partial s}\right)\right] \bar{F}(s,t) \tag{4.5}$$

The mathematical problem associated with the white noise limit is that when we substitute a parameter α by $\bar{\alpha} + \xi(t)$ in the starting master equation (2.4) the resulting stochastic transition probabilities can never be positive definite since $\xi(t)$ has unbounded realizations. As a consequence, the stochastic master equation or the associated stochastic partial differential equation for $F(s, t)$ is ill-defined and the positivity of the solution of (4.1) cannot be guaranteed in general. This problem does not occur with a dichotomic noise $\xi(t)$ which has bounded realizations. With the requirement $\bar{\alpha} \geqslant \Delta$ each step of our development is well defined. This difference between the dichotomic and white noise is explicitly seen in the requirement of positivity of the transition probabilities. The requirement found in Section 2, $Q_0 - Q_1\Delta \geqslant 0$, is always satisfied for $\bar{\alpha} \geqslant \Delta$. The requirement of positivity of (4.2) is volume dependent due to the extensivity of Q_0 and Q_1. This implies that in the white noise limit there is no value of D for which the solution of (4.1) is positive definite for all values of V. These difficulties are clearly seen in the second example of Section 3. From Eq. (3.28) it is easy to see that $\bar{P}_{st}(N)$ is positive definite if $\bar{\alpha} \geqslant \Delta$. The Gaussian white noise limit of (3.28) gives[36]

$$\bar{P}_{st}(N) = \left(\frac{\beta}{2\pi D V^2}\right)^{1/2} \frac{1}{N!} \exp\left(\frac{DV^2}{2\beta} - \frac{\alpha V}{\beta}\right) \times \int_{-\infty}^{\infty} da\, a^N \exp\left\{-\frac{\beta}{2DV^2}\left[a - \frac{V}{\beta}(\bar{\alpha} - DV)\right]^2\right\} \tag{4.6}$$

The associated generating function is

$$\bar{F}_{st}(s) = \exp\left[\frac{(DV - 2\bar{\alpha})}{2\beta} V + \left(\frac{\bar{\alpha} - DV}{\beta}\right) Vs + \frac{DV^2 s^2}{2\beta}\right] \tag{4.7}$$

From (4.7) and (2.11) it is easy to see that $\bar{P}_{st}(N)$ becomes negative for N odd if $\bar{\alpha} < DV$. The related condition for the positivity of the transition probability (4.2) is in this case $\bar{\alpha} > 2DV$. In connection with the volume dependence of these conditions it is interesting to note that in the thermodynamic limit Eq. (4.1) reduces to the correct Fokker–Planck equation which describes the system when internal fluctuations are neglected:

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x}[q_0(x) - r(x)]\, P(x, t) + D\frac{\partial}{\partial x} q_1(x) \frac{\partial}{\partial x} q_1(x)\, P(x, t) \tag{4.8}$$

Due to the difficulties discussed above it is not completely clear which is the correct physical interpretation of the results found in the white noise limit. This is even so for values of the parameters for which $W(N, N-1)$ in (4.2) is positive, because the stochastic transition probability obtained when α becomes a random parameter in (2.4) is never positive definite in this limit. The mathematical difficulties associated with the white noise limit can also be seen in a different approach in which one looks for a joint description of fluctuations in terms of a stochastic differential equation. This possibility has been partially analyzed in a different context in Ref. 44. In summary, the interpretation of the white noise limit remains as a problem with uncolved aspects requiring a deeper study which will be reported in the near future. Nevertheless we believe that in many circumstances the white noise limit is a useful approximation that gives physically sound results. It must be regarded as a calculational tool that gives correct results for the first moments of the distribution, at least for some restricted values of the parameters. For example, the relative fluctuations calculated directly from (4.6) or (4.7) coincide with the white noise limit of (3.23). Such a limit taken at the level of (3.23) makes perfect physical sense.

We finally note that there are other possible limits that may be worthwhile analyzing, for example $\lambda \to \infty$ with Δ^2 constant. In this case the difficulties mentioned above may not exist, but the physical contents of the limit should be analyzed in specific examples.

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